Lecture 6

Thursday, January 28, 2021 4:17 PM

* Prayer ....

* Spiritual thought: the Spirit leads to invention, so we need to live worthily of the Spirit.

* Answering questions ...

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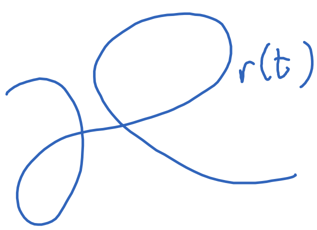
* Length of a curve $r(t)$, $a \le t \le b$



$$L = \int_a^b |r'(t)|\, dt$$

Ex $r(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2$

$$L = \int_0^2 \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = 2\sqrt{2}.$$

Note: length is an intrinsic property, independent of the way a curve is parametrized. (The distance from Provo to Salt Lake City along I-15 is the same, no matter how fast/slow you go.)

$$r(t) = \tilde{r}(\tilde{t}) = \tilde{r}(\tilde{t}(t))$$

$$L_1 = \int_a^b |r'(t)|\,dt$$

equal to each other (law of u-substitution)

$$L_2 = \int_{\tilde{a}}^{\tilde{b}} |\tilde{r}'(\tilde{t})|\,d\tilde{t} = \int_a^b |\tilde{r}(\tilde{t}(t))|\,\tilde{t}'(t)\,dt$$

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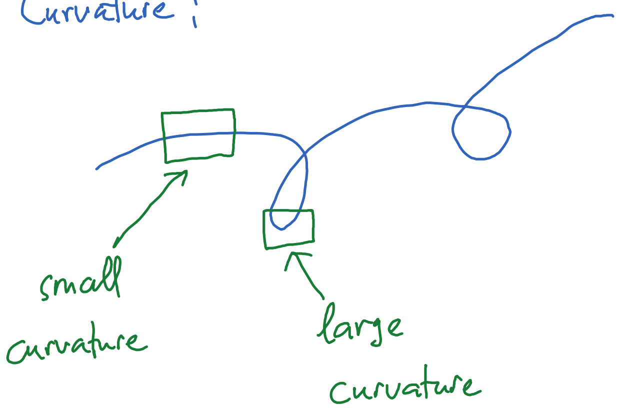
* Curvature:

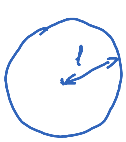


Curvature measures how fast the curve change its direction at a given point. We may be tempted to think that

$$k(t) = \left| \frac{d}{dt} r'(t) \right| = |r''(t)|.$$

This definition has two limitations:

(1) It is not independent of the parametrization of the curve.



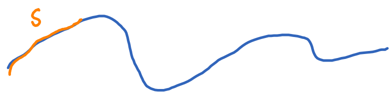
$$r_1(t): \begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = 0 \end{cases} \quad 0 \le t \le 2\pi, \qquad r_2(t): \begin{cases} x(t) = \cos 2t \\ y(t) = \sin 2t \\ z(t) = 0 \end{cases} \quad 0 \le t \le \pi$$

$$r_1''(t) = \langle -\cos t, -\sin t, 0 \rangle \quad , \quad |r_1''(t)| = 1$$

$$r_2''(t) = \langle -4\cos t, -4\sin t, 0 \rangle \quad , \quad |r_2''(t)| = 4$$

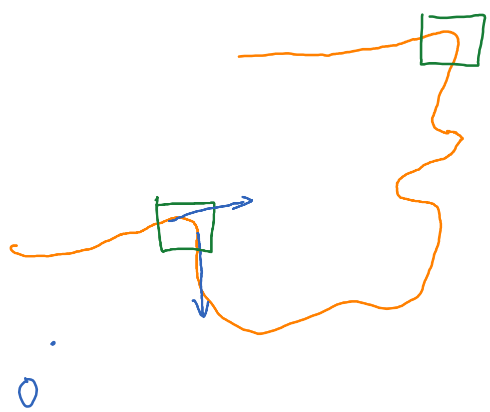
} different !

One can fix this flaw by parametrizing the curve by it <u>length</u>:



$r(s)$
$\uparrow$
length

(2) The curvature shouldn't depend on how fast you go.



$x = t \rightsquigarrow x' = 1$

$y = \sqrt{1-t^2} \rightsquigarrow y' = \dfrac{-t}{\sqrt{1-t^2}}$

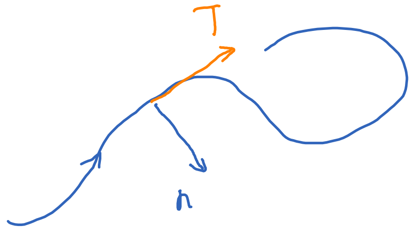
$x'^2 + y'^2 = 1 + \dfrac{t^2}{1-t^2} = \dfrac{1}{1-t^2}$

$$\kappa(s) = \left| \frac{d}{ds} \frac{r'(s)}{|r'(s)|} \right| = \left| \frac{dT}{ds} \right|$$

$$\rightsquigarrow \quad \kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$

Alternatively,

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$



Normal vector: (principle)

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

Binormal vector:

$$B(t) = T(t) \times N(t)$$

Torsion:

$$\tau = \frac{[r'(t) \times r''(t)] \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

$$\tau = -\frac{dB}{ds} \cdot N$$